Combinatorics in Banach space theory

PROBLEMS (Part 5)*

● **PROBLEM 5.1.** Let Σ be the σ-algebra of all Borel subsets of $[0,1]$. Define a vector measure $\mu\colon \Sigma \to L_1[0,1]$ by $\mu(E) = \mathbb{1}_E$. Verify that μ is non-atomic and has bounded variation. Next, show that the range of μ is closed, but is neither convex nor compact.

Remark. This exercise shows that the assumptions upon the target space of a given vector measure in Theorems 11.10 and 12.11 are not superfluous. In fact, $L_1[0,1]$ is probably the most natural example of a Banach space which simultaneously violates all the three conditions (reflexivity, being a separable dual space and B-convexity) appearing in those theorems.

● **PROBLEM 5.2.** Here is another counterexample which shows that being WCG is not a 3SP property (recall Proposition 8.6). Let $D[0,1]$ be the subspace of $L_\infty[0,1]$ consisting of all real-valued functions that are right continuous and have finite left limits at every point of $[0,1]$.

(i) Show that $D[0,1]$ is closed in $L_\infty[0,1]$ and hence it is a Banach space when equipped with the supremum norm.
(ii) Of course, $C[0,1]$ is embedded (via the inclusion operator) in $D[0,1]$. Show that the quotient space $D[0,1]/C[0,1]$ is isometrically isomorphic to $c_0[0,1] \simeq c_0(\mathfrak{c})$.

Hint. It is useful to define a quotient operator $Q\colon D[0,1] \to c_0[0,1]$ such that $\operatorname{dist}(f, C[0,1]) = \|Q(f)\|$ for every $f \in D[0,1]$.

Remark. $D[0,1]$ is isomorphic to $C(\mathbb{A})$, the Banach space of real continuous functions on the *Alexandrov double-arrow space* $\mathbb{A} = [0,1] \times \{0,1\}$ topologised by the order topology generated by the lexicographic order. It may be shown that this space is not normal in its weak topology (see [FHH, Theorem 14.39]), thus it cannot be weakly Lindelöf (a topological space is Lindelöf if every open cover admits a countable subcover), because every regular Lindelöf topological space is normal (for weak topologies on Banach spaces being Lindelöf and being normal are equivalent). However, by the Preiss–Talagrand theorem (see [FHH10, Theorem 14.31]), every WCG Banach space is weakly Lindelöf and therefore the exact sequence $0 \to C[0,1] \to D[0,1] \to c_0(\mathfrak{c}) \to 0$ shows that being WCG is not a 3SP property.

● **PROBLEM 5.3.** Show that if X is a B-convex Banach space, then X^{**} is B-convex too.

Hint. You shall use the characterisation of B-convexity given in Theorem 12.3 combining it with the fact that X^{**} is finitely representable in X (see Definition 12.2). The last statement follows from the Principle of Local Reflexivity (see [AK06, §11.2]).

● **PROBLEM 5.4.** Let X and Y be normed spaces. Assume $f\colon X \to Y$ is a *quasi-linear* map, that is, $f(\lambda x) = \lambda f(x)$ and $\|f(x+y) - f(x) - f(y)\| \leqslant c\cdot(\|x\| + \|y\|)$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$, where $c < \infty$ is a constant. Show that for all $x_1, \ldots, x_n \in X$ we have

$$\left\| f\left(\sum_{j=1}^{n} x_j\right) - \sum_{j=1}^{n} f(x_j) \right\| \leqslant c \sum_{j=1}^{n} j\|x_j\|.$$

*Evaluation: ● =2pt, ● =3pt, ● =4pt

PROBLEM 5.5. Let $\mathfrak{U} = (A_1, \dots, A_m)$ be a family of finite sets. By a *transversal* of $\mathfrak{U}$ we mean any injective map $f\colon [m] \to \bigcup_{j=1}^m A_j$ such that $f(j) \in A_j$ for each $j \in [m]$ (that is, any one-to-one selection). Prove the *Hall marriage lemma* which says that there exists a transversal for $\mathfrak{U}$ if and only if $\mathfrak{U}$ satisfies the following Hall's condition:

$$\left| \bigcup_{j\in J} A_j \right| \geqslant |J| \quad \text{for every } J \subset [m].$$

Hint. There are various proofs of this classical result in the literature. For example, you may like to use the following approach. Suppose Hall's condition is valid and in order to prove that $\mathfrak{U}$ has a transversal (the converse implication is obvious) define $\mathfrak{V} = (B_1, \dots, B_m)$ to be the collection of minimal sets $B_j \subset A_j$ ($j \in [m]$) for which Hall's condition survives. Show that each B_j is a singleton.

Remark. Hall's lemma, proved in 1935, has found several astonishing application in mathematical analysis. For instance, it was used by G.E. Bredon to give a relatively short construction of Haar measure on topological groups. We shall see the power of Hall's lemma in the proof of the Kalton–Roberts theorem.

PROBLEM 5.6. Let $\mathfrak{U} = (A_1, \dots, A_m)$ be a family of finite sets and let $d \in \mathbb{N}$. Suppose each member of $\mathfrak{U}$ has at least d elements and none of those elements appears in more than d sets from $\mathfrak{U}$. Prove that $\mathfrak{U}$ has a transversal.

Hint. Verify Hall's condition. Pick any k members of $\mathfrak{U}$ and write down all their elements allowing repetitions. How many pairwise different elements must appear?

PROBLEM 5.7. Let $\mathscr{F}$ be a finite set algebra. For every function $\nu\colon \mathscr{F} \to \mathbb{R}$ define $V(\nu) = \max_{A,B\in\mathscr{F}}(\nu(A) - \nu(B))$. Let also $\mathcal{M}$ be the set of all real valued, finitely additive measures on $\mathscr{F}$. Show that for every $\nu\colon \mathscr{F} \to \mathbb{R}$ there exists $\mu \in \mathcal{M}$ such that

$$V(\nu - \mu) = \inf\Big\{V(\nu - \lambda)\colon \lambda \in \mathcal{M}\Big\}.$$

PROBLEM 5.8. Let K be a compact Hausdorff space and assume that a (uniformly) bounded sequence $(f_n)_{n=1}^\infty \subset C(K)$ converges pointwise to $f \in C(K)$. Prove that for every $\varepsilon > 0$ there exist $n_1, \dots, n_k \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_k \in [0,1]$ such that $\sum_{j=1}^k \lambda_j = 1$ and

$$\left\| f - \sum_{j=1}^k \lambda_j f_{n_j} \right\| \leqslant \varepsilon.$$

Hint. Assume, with no loss of generality, that $f = 0$ and $\|f_n\| \leqslant 1$ for each $n \in \mathbb{N}$. For any given $\varepsilon > 0$ and $x \in K$ set $G_x = \{n \in \mathbb{N}\colon |f_n(x)| \geqslant \varepsilon/2\}$ and define a hereditary family $\mathcal{G} \subset \mathcal{F}\mathbb{N}$ by

$$\mathcal{G} = \{G \in \mathcal{F}\mathbb{N}\colon G \subset G_x \text{ for some } x \in K\}.$$

Use the contrapositive version of Pták's combinatorial lemma by verifying that $\mathcal{G}$ fails to satisfy the assertion of Lemma 9.7.

Remark. This is a specialised version of the classical Mazur theorem (see [Rud91,Theorem 3.13]) which says that for every weakly convergent sequence $(x_n)_{n=1}^\infty$ in a Banach space there exists a strongly convergent sequence all of whose elements are convex combinations of x_n's.

PROBLEM 5.9. Prove that ℓ_∞^* is not w^*-sequentially separable.

Hint. ℓ_∞ is a Grothendieck space.

Remark. Note that ℓ_∞^* is obviously w^*-separable, which follows from Goldstine's theorem, as ℓ_∞^* is the bidual of the (norm) separable space ℓ_1.

PROBLEM 5.10. Prove that a complemented subspace of a Banach space with the Dunford–Pettis property (see Definition 5.3) also has the Dunford–Pettis property. Use this fact to show that $L_1[0,1]$ does not contain a complemented subspace isomorphic to ℓ_p, for every $p \in (1,\infty)$.

Hint. For the first assertion use the characterisation of the Dunford–Pettis property stated in Lemma 5.4. Next, recall that $L_1[0,1]$ has the Dunford–Pettis property (see Remark 5.6).

PROBLEM 5.11. Let $\mathscr{F}$ be an algebra of subsets of Ω (equivalently, any Boolean algebra) and $\mathcal{A}$ be any non-empty subfamily of $\mathscr{F}$. We define the *intersection number* of $\mathcal{A}$, denoted $\mathsf{I}(\mathcal{A})$, as the largest $\delta \geqslant 0$ such that for every finite sequence $(A_j)_{j=1}^n \subset \mathcal{A}$ (repetitions are allowed) there is a set $J \subset [n]$ such that $|J| \geqslant \delta \cdot n$ and $\bigcap_{j\in J} A_j \neq \varnothing$. In other words,

$$\mathsf{I}(\mathcal{A}) = \inf\Big\{\frac{1}{n}\sup_{x\in\Omega}\sum_{j=1}^n \mathbb{1}_{A_j}(x)\colon (A_j)_{j=1}^n \subset \mathcal{A}\Big\}.$$

Show that for every finitely additive measure $m\colon \mathscr{F} \to [0,1]$ with $m(\Omega) = 1$ we have $\inf_{A\in\mathcal{A}} m(A) \leqslant \mathsf{I}(\mathcal{A})$.

PROBLEM 5.12. Let $\mathscr{F}$ be an algebra of subsets of Ω and $\mathcal{A}$ be any non-empty subfamily of $\mathscr{F}$. We define the *covering index* of $\mathcal{A}$, denoted $\mathsf{C}(\mathcal{A})$, as the largest $\delta \geqslant 0$ for which there exists a finite sequence $(A_j)_{j=1}^n \subset \mathcal{A}$ (possibly with repetitions) such that $t\mathbb{1}_\Omega \leqslant \frac{1}{n}\sum_{j=1}^n \mathbb{1}_{A_j}$. In other words,

$$\mathsf{C}(\mathcal{A}) = \sup\Big\{\frac{1}{n}\inf_{x\in\Omega}\sum_{j=1}^n \mathbb{1}_{A_j}(x)\colon (A_j)_{j=1}^n \subset \mathcal{A}\Big\}.$$

Let

$$\gamma(\mathcal{A}) = \inf\Bigg\{\sum_{A\in\mathcal{A}} x_A\colon \sum_{A\in\mathcal{A}} x_A\mathbb{1}_A \geqslant \mathbb{1}_\Omega \text{ and } x_A \geqslant 0 \text{ for all } A\in\mathcal{A}\Bigg\}.$$

Show that $\mathsf{C}(\mathcal{A}) = \gamma(\mathcal{A})^{-1}$.

PROBLEM 5.13. Let $\mathcal{A}$ be a non-empty subfamily of a given set algebra $\mathscr{F}$. Let also $\mathcal{A}^{\mathsf{c}}$ be the class of complements of all members of $\mathcal{A}$. Prove the duality relation $\mathsf{I}(\mathcal{A}) + \mathsf{C}(\mathcal{A}^{\mathsf{c}}) = 1$.

PROBLEM 5.14. Let $\mathscr{F}$ be a set algebra and $\varnothing \neq \mathcal{A} \subset \mathscr{F}$. Show that

$$\mathsf{I}(\mathcal{A}) = \sup\Big\{\mathsf{I}(\mathcal{B})\colon \mathcal{B} \neq \varnothing \text{ is a finite subset of } \mathcal{A}\Big\}.$$

PROBLEM 5.15. Let $\Omega = [4]$, $\mathscr{F} = \mathcal{P}\Omega$ and consider the collection

$$\mathcal{A} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}.$$

Show that if $A_1,\ldots,A_n$ are distinct members of $\mathcal{A}$ then there exists a set $J \subset [n]$ such that $|J| \geqslant \frac{2}{3}n$ and $\bigcap_{j\in J} A_j \neq \varnothing$, however the intersection number $\mathsf{I}(\mathcal{A})$ equals $\frac{3}{5}$.

Hint. For showing that $\mathsf{I}(\mathcal{A}) = \frac{3}{5}$ use Kelley's theorem in the following form:

$$\mathsf{I}(\mathcal{A}) = \max_{m\in\mathcal{M}} \inf_{A\in\mathcal{A}} m(A),$$

where $\mathcal{M}$ stands for the family of all finitely additive measures $m\colon \mathscr{F} \to [0,1]$ with $m(\Omega) = 1$.

● **PROBLEM 5.16.** Prove that a Banach space X is B-convex if and only if so is X^*.

Hint. It suffices to show that if X^* is B-convex, then so is X, because having this implication we may simply appeal to the assertion of Problem 5.3 to get the reverse one. So, suppose X is not B-convex. Then, according to Theorem 12.3, X contains finite-dimensional spaces E_n's which are 2-isomorphic to ℓ_1's (i.e. $d_{\mathsf{BM}}(E_n, \ell_1^n) \leqslant 2$). For any $n \in \mathbb{N}$ pick a norm one surjective operator $Q_n\colon \ell_1 \to \ell_\infty^n$ and $x_1, \ldots, x_n \in \ell_1$ with $Q_n(x_j) = e_j$ for each $j \in [n]$. In view of Lemma 12.7 (ℓ_1 in a $\mathscr{L}_1$-space), there is $N \geqslant n$ and an N-dimensional subspace F_N of ℓ_1 which contains $\{x_1, \ldots, x_n\}$ and is 2-isomorphic to ℓ_1^N. By considering a suitable adjoint operator, try to produce a subspace of X^* that is 4-isomorphic to ℓ_1^N.

● **PROBLEM 5.17.** Prove the following statement called *root lemma* or Δ-*system lemma*: If $\mathcal{A}$ is an uncountable family of finite sets, then there exists an uncountable subfamily $\mathcal{B}$ of $\mathcal{A}$ and a finite (possibly empty) set S such that $A \cap B = S$ for all $A, B \in \mathcal{B}$ with $A \neq B$.

Hint. With no loss generality we may assume that $|\mathcal{A}| = \aleph_1$ and all members of $\mathcal{A}$ are finite subsets of the ordinal interval $[0, \omega_1]$. Show that for some $n \in \mathbb{N}$ the collection $\mathcal{A}_n := \{A \in \mathcal{A}\colon |A| = n\}$ is uncountable and $\sup(\bigcup_{A\in\mathcal{A}_n} A) = \omega_1$. For each $A \in \mathcal{A}_n$ write $A = \{A(1), \ldots, A(n)\}$ with $A(1) < \ldots < A(n)$ and define $p \in [n]$ to be the least integer satisfying $\sup\{A(p)\colon A \in \mathcal{A}_n\} = \omega_1$. Now, let

$$\alpha_0 = \begin{cases} 0, & \text{if } p = 1 \\ 1 + \sup\limits_{A\in\mathcal{A}_n} A(p-1), & \text{if } p > 1. \end{cases}$$

and use transfinite induction to produce a sequence $(A_\nu)_{0\leqslant\nu<\omega_1} \subset \mathcal{A}_n$ such that

$$A_\nu(p) > \max\Big\{\alpha_0, \sup\{A_\alpha(p)\colon \alpha < \nu\}\Big\} \quad \text{for every } \nu \in [0, \omega_1).$$

Finally, show that the collection $\mathcal{B}_1 := \{A_\nu\colon 0 \leqslant \nu < \omega_1\}$ contains the desired subfamily $\mathcal{B}$.